Titolo del corso:

Homology of braid groups and mapping class groups (Omologia dei gruppi di trecce e dei gruppi modulari)

Docente:

Andrea Bianchi

Membro del collegio proponente:

Giovanni Mongardi

Ore frontali di lezione:

20

Periodo di lezione

Novembre 2026-febbraio 2027

Settori disciplinari del corso: Geometria (Mat03 – MATH02/b)

Tipologia di corso:

base

Modalità di verifica dell'apprendimento:

Esame a seminario con domande

Abstract del corso:

In this course we will introduce two of the most important characters in geometric topology: braid groups and mapping class groups, together with their topological counterparts: configuration spaces and moduli spaces of surfaces. We will focus on the problem of computing the homology of these groups/spaces, and in particular we will prove homology stability in both cases, and identify, more or less explicitly, the stable homology. While for braid groups we will be able to compute also the unstable homology, for mapping class

groups many questions remain open, and we will outline a possible strategy to study the unstable homology via amalgamated Hurwitz spaces. Along the course a few key concepts in topology, such as basic group theory, fibre bundles, higher homotopy groups, spectral sequences, simplicial methods, basics of homological algebra, basics of category theory will be recalled, to different extent as needed, so that the student who has some knowledge of algebraic topology, including the notions of fundamental group, singular homology, cell decompositions of spaces, smooth surfaces, will be able to start attending the course.

Programma del corso:

1) Definition of configuration spaces and of braid groups; a few explicit computations; Artin's presentation; relation to symmetric groups.

2) Fadel--Neuwirth fibre bundles and asphericity of Conf_n(R^2), and more generally of Conf_n(S) for S an aspherical surface (classification of surfaces...).

3) Definition of mapping class groups of surfaces; examples of mapping classes, Dehn twists; braid relation; generation by Dehn twists; results about presentation of mapping class groups; example Br_3=Gamma_1,1

4) Recollections on group homology and group cohomology; Eilenberg--Maclane spaces and classifying spaces of discrete groups; examples of computations (free groups, free abelian groups, surface groups, third braid group).

5) Basics of Teichmueller theory: Riemann surfaces, Teichmueller space, homeomorphism with a Euclidean ball (statement), action of the mapping class group, stabilisers are finite; moduli spaces as classifying spaces of mapping class groups.

6) Homology stability for sequences of spaces; how it arises in the case of the sequence of classifying spaces of braid groups and that of mapping class groups.

7) A general strategy to prove homology stability via spectral sequences and using highly connected semisimplicial spaces: recollections on spectral sequences; the spectral sequence associated with a semisimplicial space.

8) Arc complexes and their high connectivity; proof of homology stability for braid groups and for mapping class groups.

9) Topological monoids and their classifying spaces; (strict) topological monoid versions of configuration spaces and of moduli spaces; example of loop spaces as non-strict topological monoids, and definition of E_1-structures; statement of the group-completion theorem.

10) Identification of the group completions of the monoids of configuration spaces and of moduli spaces (statements); statement of the Mumford conjecture, now a theorem of Madsen--Weiss.

11) Computation of the full homology groups of braid groups: recollections on Poincare'--Lefschetz duality; Fuchs' cell stratification on $\mathrm{Conf}_n(\mathbb{R}^2)$; interpretationas a bar complex; computation of Tor/Ext groups over 1-generated algebras over a field.

12) Hurwitz spaces and their relation to moduli spaces of Riemann surfaces; amalgamated Hurwitz spaces give a model of the homotopy type of $M_{g,n}$ for $n \geq 1$; description of the cell stratification of Hurwitz spaces.